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Abstract

It is shown that highly localized solitons can be created in lower dimensional Bose–Einstein condensates (BECs), trapped in a regular harmonic trap, by temporally varying the trap frequency. A BEC confined in such a trap can be effectively used to construct a pulsed atomic laser emitting coherent atomic wave packets. In addition to having a complete control over the spatio-temporal dynamics of the solitons, we can separate the equation governing the Kohn mode (centre of mass motion). We investigate the effect of the temporal modulation of the trap frequency on the spatio-temporal dynamics of the bright solitons and also on the Kohn mode. The dynamics of the solitons and the variations in the Kohn mode with time are compared with those in a BEC confined in a trap with unmodulated trap frequency.

PACS numbers: 03.75.−b, 05.45.Yv, 03.75.Pp

(Some figures in this article are in colour only in the electronic version)

1. Introduction

The creation of the Bose–Einstein condensate (BEC) [1] has opened up an area with a potential for enormous technological application. On one hand, active research, both theoretical and experimental, is being carried out to understand the properties of this new phase of matter, and on the other hand practical applications of the condensate are being developed. The fact that a BEC constitutes coherent matter waves analogous to the coherent light waves makes it the most suitable candidate for the construction of an atom laser. The first atomic laser was constructed in 1997 in MIT [2] and other experimental results were reported in [3, 4]. An atom laser can be used to develop accurate atomic clocks [5], gyroscopes for ultra-precision navigation [6] and will also help in the development of coherent atom optics [7, 8], atom interferometry [9, 10] and atomic transport [11]. The realization of bright solitons, soliton trains [12–18] and grey solitons [19] in BECs has given further impetus to this line of research. Furthermore, a

lot of work on the stability of the solitons, control and manipulation of these states have been carried out [20–23], which is essential for the development of an atom laser and other useful applications of the condensates.

Solitons are ubiquitous in nature and have a unique property that they do not disperse even after travelling long distances. Optical solitons have been successfully used for the purpose of telecommunications [24]. A bright soliton is a bunch of atoms moving together, and its density is more compared to the background. It has been shown in [14, 25–27] that solitonic pulses can be generated in a BEC trapped in an elongated harmonic trap, by simultaneously changing the scattering length and by making the trap expulsive. Generation of soliton pulses for a pulsed atom laser using the quantum state transfer property was discussed by Liu *et al* [28], and the dynamics of an atom laser were studied in [29–31].

It is a well-known fact that the Gross–Pitaevskii (GP) equation describes the dynamics of a BEC at the mean-field level. An analytical method to solve the GP equation for a BEC trapped in an elongated harmonic trap was presented in [32]. Here, both the scattering length and the trap frequency are functions of time, and it was seen that the phase of the solitonic solution got related to the trap frequency in a non-trivial way. This relation was exploited to obtain a complete control over the spatio-temporal dynamics of the condensate, in addition to obtaining exact expressions for the solutions. Moreover, it provided one with the freedom to vary the trap frequency and also to tune the atomic scattering length. Thus, this method allowed one to study the effects of variations in the trap frequency on the dynamics of the BEC theoretically and also to see how the nonlinearity changes with time. We want to point out here that this method works even when the trap frequency is independent of time, but our goal is to introduce certain temporal variations in the trap frequency and study their effects on the soliton dynamics. This becomes all the more interesting because of the fact that the present technology allows construction of confining potentials of arbitrary shapes in the laboratory [33, 34]. Added to this, the lower dimensional BECs offer a lot of flexibility in terms of the control and fine tuning of the experimental parameters, making these systems interesting to analyse. In our earlier study, we have used this method to study the effect of sudden temporal changes in the trap frequency, of an expulsive harmonic trap, which impart a sudden strong jolt to the system [35]. For some of these modulations, highly localized solitons were observed. Similar studies have been done in [36].

In the present paper, we study the spatio-temporal dynamics of a BEC confined in a regular harmonic trap, whose trap frequency is modulated using an oscillatory function which vanishes asymptotically. Such a variation of the frequency results in a slow vibration of the trap, which die down with time. Hence after a time *t*, the trap frequency is almost a constant and the system behaves as if there is no modulation in the frequency. Thus, it becomes natural to compare the soliton dynamics in this case with the one, where the trap frequency is a constant in time and study how an initial variation in the frequency affects the soliton dynamics in terms of localization and amplification.

We have observed that these modulations result in highly compressed solitons, with very high localization and amplification, as compared to the solitons created in a trap with a constant trap frequency. Since in the context of pulsed atomic lasers, such solitons are highly desirable, our study concentrates on the generation of localized bright solitons. We have seen that these solitons are stable in certain ranges of *t*, and the evolution of the bright solitons in these specific regions is investigated. We also point out here that in this method of obtaining solutions of the GP equation, one of the consistency conditions obtained governs the centre of mass (COM) motion. From Kohn's theorem [37], we know that for a BEC confined in a harmonic trap with constant frequency, the COM oscillates with the frequency of the trap, giving rise to a resonance known as the Kohn mode. It is also known that when the trap frequency varies with

time the COM motion becomes nontrivial [38]. In our case, we find that the COM oscillates with the trap frequency, but due to the presence of the trap modulation, its amplitude decreases with time.

2. Solutions of the NLSE

The dynamics of a dilute Bose gas trapped in a cylindrical harmonic trap with a time-dependent confinement in the z direction given by $V = m'\omega_\perp^2(x^2 + y^2)/2 + m'\omega_0^2(t)z^2/2$, is governed by the GP equation. This equation is reduced to the quasi-one-dimensional nonlinear Schrödinger equation (NLSE), with a tight confinement in the z direction ($\omega_\perp \gg \omega_0$), using a Gaussian trial wavefunction [32, 39, 40],

$$\iota \partial_t \psi = -\tfrac{1}{2}\partial_{zz}\psi + \gamma(t)|\psi|^2\psi + \tfrac{1}{2}M(t)z^2\psi, \tag{1}$$

where $\gamma(t) = 2a_s(t)/a_B$, $M(t) = \omega_0^2(t)/\omega_\perp^2$ and $a_\perp = (\hbar/m'\omega_\perp)^{1/2}$. Here $\gamma(t)$ is the nonlinear coupling, which is attractive if $\gamma(t) < 0$ and repulsive if $\gamma(t) > 0$. $a_s(t)$ is the scattering length and a_B is Bohr's radius. Note that $M(t) > 0$ corresponds to the regular oscillator and $M(t) < 0$ corresponds to the expulsive oscillator. The exact solutions of the NLSE are obtained by using the ansatz solution

$$\psi(z, t) = \sqrt{A(t)}F\{A(t)[z - l(t)]\}\exp[\iota\Phi(z, t)], \tag{2}$$

where

$$\Phi(z, t) = a(t) - \tfrac{1}{2}c(t)z^2 \tag{3}$$

with $a(t) = a_0 + \frac{\lambda-1}{2}\int_0^t A^2(t')\,dt'$, in which a_0 and λ are constants. Here, $A(t)$ gives the amplitude and $l(t)$ gives the location of the centre of mass of the soliton, respectively. Substitution of $\psi(z, t)$ in the NLSE gives

$$A(t) = A_0 \exp\left(\int_0^t c(t')\,dt'\right), \qquad \gamma(t) = \gamma_0\frac{A(t)}{A_0}, \tag{4}$$

and

$$\frac{dl(t)}{dt} + c(t)l(t) = 0, \tag{5}$$

which govern the control parameters, namely the amplitude, nonlinearity and the location of the COM, respectively. Here, A_0, γ_0 and l_0 are constants of integration, where $A_0 > 0$ and γ_0, l_0 are real numbers. Use of the above consistency conditions reduces the NLSE to the elliptic differential equation for F:

$$F''(T) - \lambda F(T) + 2\kappa F^3(T) = 0 \tag{6}$$

with $\kappa = -\frac{\gamma_0}{A_0}$, $T = A(t)(z - l(t))$ and the differentiation is with respect to T. Solutions of this equation are the 12 Jacobi elliptic functions [41]. Specifically, in the attractive nonlinear regime ($\gamma_0 < 0$), the solution $F(T) = \text{cn}(T/\tau_0, m)$ of (2) describes soliton trains, with $\tau_0^2 = -A_0 m/\gamma_0$, $\lambda = (1 - 2m)/\tau_0^2$. In the limit $m \to 1$, $\text{cn}(T, m) \to \text{sech}(T)$, and this same solution describes a bright soliton given by

$$\psi(z, t) = \sqrt{A(t)}\,\text{sech}(T/\tau_0)\exp\left[\iota a(t) - \frac{\iota}{2}c(t)z^2\right], \tag{7}$$

with $\tau_0^2 = -A_0/\gamma_0$. Similarly, the solution $F(T) = \text{sn}(T/\tau_0, m)$ of (2) describes soliton trains in the repulsive nonlinear regime ($\gamma_0 > 0$) with $\tau_0^2 = A_0/\gamma_0$, $\lambda = -(1 + m)/\tau_0^2$. The dark soliton is obtained in the limit $m \to 1$ as $\text{sn}(T, m) \to \text{tanh}(T)$ and is given by

$$\psi(z, t) = \sqrt{A(t)}\,\text{tanh}(T/\tau_0)\exp\left[\iota a(t) - \frac{\iota}{2}c(t)z^2\right]. \tag{8}$$

From (4) and (5), we see that the control parameters are non-trivially related to $c(t)$ appearing in the phase of the wavefunction. Pointing out here that (5) describes the motion of the COM and for a given trap modulation, this equation can be used to study the effect of modulation on the Kohn mode. In addition to the above consistency conditions, it is also found that $c(t)$ satisfies the Riccati equation

$$\frac{dc(t)}{dt} - c^2(t) = M(t), \tag{9}$$

which can be mapped to a second-order differential equation using the change of the variable

$$c(t) = -\frac{d}{dt} \ln[\phi(t)]. \tag{10}$$

The new equation,

$$-\phi''(t) - M(t)\phi(t) = 0, \tag{11}$$

is in the form of the Schrödinger equation, if we write the ratio of the trap frequencies as $M(t) = M_0 + V(t)$. Here, $V(t)$ is like a potential with the eigenvalue M_0, and this allows us to introduce temporal modulations in the trap frequency. Thus, taking advantage of the form of (11), we can modify the temporal profile of the trap frequency using potentials for which the Schrödinger equation is exactly solvable. By substituting the known solutions $\phi(t)$ of (11), one can obtain $c(t)$ which when used in (4)–(5) gives the consistency conditions in terms of $\phi(t)$ as follows:

$$A(t) = A_0 \frac{\phi(0)}{\phi(t)}, \qquad \gamma(t) = \gamma_0 \frac{\phi(0)}{\phi(t)}, \qquad l(t) = l_0 \frac{\phi(t)}{\phi(0)}. \tag{12}$$

Thus, one obtains exact expressions for the control parameters, which allow us to write the complete solution of the NLSE. These solutions can now be used to study and control the spatio-temporal dynamics of the solitons.

3. Soliton dynamics

As mentioned in the introduction, the present paper looks at the soliton dynamics when the trap frequency of the regular harmonic trap is modulated by an oscillatory function which vanishes for large t. One comes across such analytically tractable potentials with positive energy states, in the study of bound states in continuum (BIC), which were first predicted by von Neumann and Wigner [42]. In a study connected with such states, Pappademos *et al* [43], have constructed BIC by deforming the free-particle potential. This deformation resulted in oscillatory potentials, vanishing asymptotically and for which exact solutions were known. Thus, use of these potentials as the modulation functions in (11) introduces vibrations in the trap which are initially large and then gradually taper down with time. Hence for large t, the effect of the modulations on the trap becomes negligible, thus making it very natural to compare this case with that where the trap frequency is constant in time. For the present study, we will consider two of the deformed potentials discussed in [43] as $V(t)$ in (11) and use their eigenfunctions to obtain the solitonic solutions.

First, we study the soliton dynamics in the BEC whose trap frequency is modulated by potential $V(t) \equiv V(t, \lambda')$, $t > 0$:

$$V(t, \lambda') = \frac{32 M_0 \cos^4(\sqrt{M_0}\, t)}{D_0^2(t)} + \frac{8 M_0 \sin(2\sqrt{M_0}\, t)}{D_0(t)}, \tag{13}$$

where $D_0(t) = 2\sqrt{M_0}\, t + \sin(2\sqrt{M_0}\, t) + 4\sqrt{M_0}\, \lambda'$, $\sqrt{M_0} > 0$ and $\lambda' > 0$. From the above expression, we can see that this modulational potential vanishes asymptotically. The effect
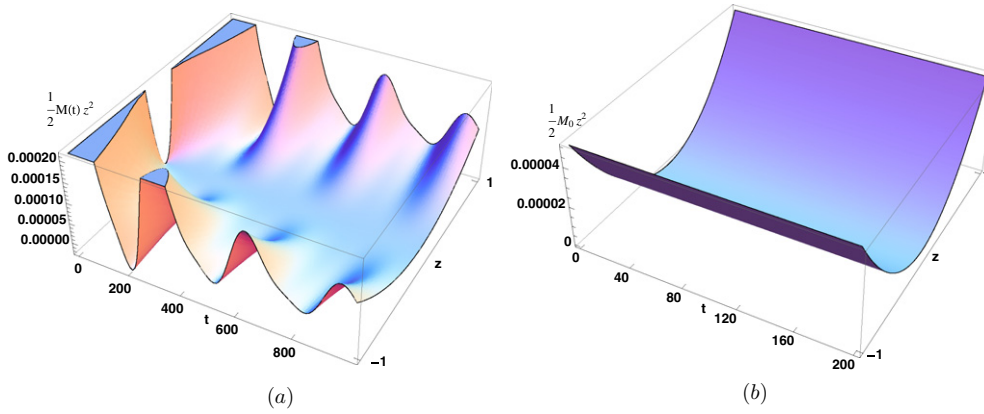
Figure 1. Plots of the confining traps with $M_0 = 0.01$, $\lambda = 1$. (*a*) Modulated trap potential and (*b*) unmodulated trap potential.

of the modulation on the harmonic trap in the NLSE can be seen in figure 1(*a*). Here, the modulated trap potential, $\frac{1}{2}(M_0 + V(t, \lambda'))z^2$, is plotted for increasing time, and we see that the modulation introduces oscillations in the trap, which are strong initially but decay gradually with time. We can see from (13) that the larger value of M_0, the larger will be the frequency of these oscillations. In addition, for higher values of M_0 and λ', the oscillations decay faster, i.e. $V(t, \lambda') \to 0$. Since $V(t, \lambda') \to 0$ for increasing t, we have $M(t) \approx M_0$, a constant in (11), which is nothing but the free-particle Schrödinger equation. Thus, we can see that this example provides an ideal situation where we can compare dynamics in a modulated trap to that in an unmodulated trap, as the former system tends to the latter in certain limits. Thus for comparison, we also give the plot of the unmodulated trap, which shows that the confining potential, $\frac{1}{2}M_0 z^2$, is smooth.

Substituting the known solution of (11) with the above potential,

$$\tilde{\phi}_1(t, \lambda') = \frac{4\sqrt{M_0} \cos(\sqrt{M_0}\, t)}{D_0(t)}, \qquad (14)$$

with the energy eigenvalue M_0, in (12), we get the expressions for amplitude and nonlinearity as

$$A(t) = A_0 \left[\frac{D_0(t)}{4\lambda'\sqrt{M_0}} \right] \sec(\sqrt{M_0}\, t), \qquad (15)$$

$$\gamma(t) = \gamma_0 \left[\frac{D_0(t)}{4\lambda'\sqrt{M_0}} \right] \sec(\sqrt{M_0}\, t) \qquad (16)$$

and the COM motion is given by

$$l(t) = l_0 \left[\frac{4\lambda'\sqrt{M_0}}{D_0(t)} \right] \cos(\sqrt{M_0}\, t). \qquad (17)$$

Since the goal of this paper is to construct localized atomic wave packets, we concentrate on the spatio-temporal dynamics of the bright soliton alone. Thus, the substitution of the above equations in (7) gives the bright soliton solution.

In order to keep our study experimentally relevant, we analyse the soliton dynamics in the range of parameters in which the system is effectively one dimensional. The parameter
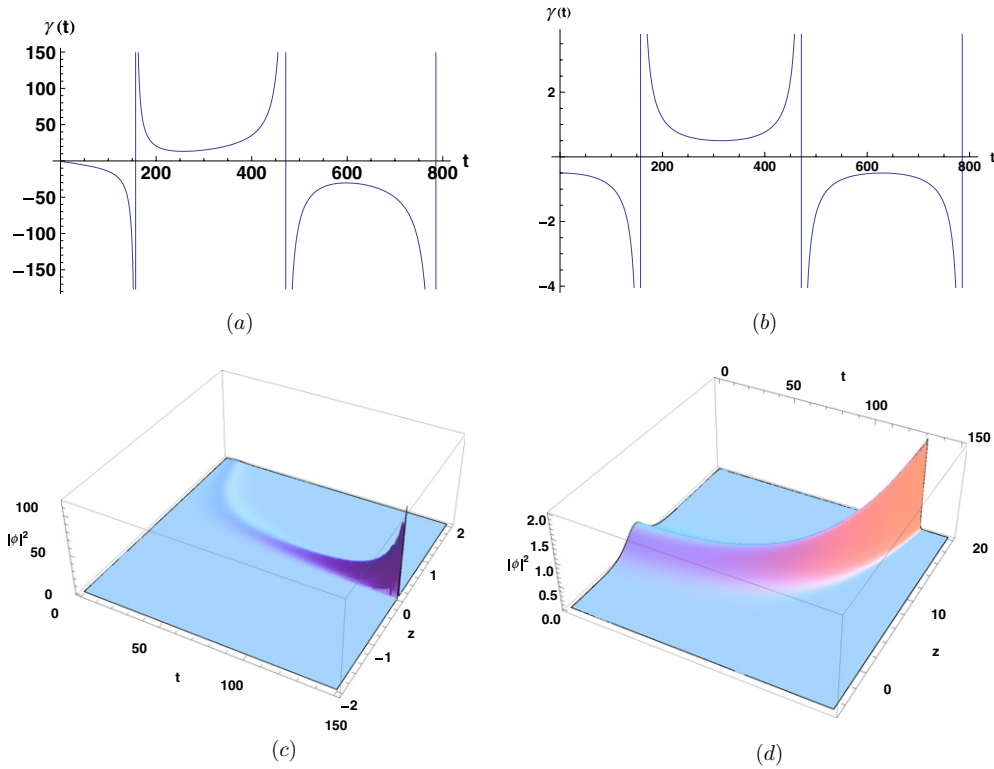
Figure 2. The parameter values are $\gamma_0 = -0.5$, $M_0 = 0.0001$, $A_0 = 0.5$, $l_0 = 5$, $\tau_0 = 1$ and $\lambda = 5$. (*a*) The nonlinearity $\gamma(t)$ in a modulated trap, (*b*) the nonlinearity $\gamma(t)$ in an unmodulated trap, (*c*) bright soliton in a modulated trap and (*d*) bright soliton in an unmodulated trap.

values used in one of the experiments involving the cigar-shaped condensate of ^7Li are $N \approx 10^3$, $\omega_\perp = 2\pi \times 700$ Hz and $\omega_0 = 2\pi \times 7$ Hz [14], where $\omega_0/\omega_\perp \approx 10^{-3}$, satisfying the condition $\omega_0/\omega_\perp < 1$ needed for the creation of a cigar-shaped BEC. Therefore, in our study, we take $M_0 = 0.0001$ and $M(t)$ in (1) is very small and of the order 10^{-3}.

It can be seen from figure 2(*a*) that the nonlinearity changes the sign periodically and discontinuously. Since the emphasis is on the study of bright solitons, we choose the range of t where the nonlinearity is negative and continuous. We point out here that in the 1D approximation, the two-body interaction energy is much less than the kinetic energy in the transverse direction, i.e. $\epsilon^2 \sim N|a_s|/a_0 \ll 1$ [16], where N denotes the number of atoms in the condensate and $a_0 = (\hbar/m'\omega_0)^{1/2}$. The above condition translates to $\gamma(t) \ll 2a_0/a_B N$ for our equations which gives a constraint on the allowed values of $\gamma(t)$. In our case, as the effect of the modulation is initially more, we study the evolution of the soliton in the first interval of time, where the nonlinearity is negative and continuous. Moreover, the time of evolution was taken such that at all times the nonlinearity obeyed the above constraint. In this range, the evolution of matter wave density $|\psi(z, t)|^2$ is showed for varying z and t in figure 2(*c*). We can see that the bright soliton is localized and with time gets compressed. For comparison, we plot the nonlinearity and the dynamics of the bright soliton for the case where the trap frequency is constant, i.e., $V(t, \lambda') = 0$ in figures 2(*b*) and (*d*) respectively. As in this case (11) is nothing but the Schrödinger equation for a free particle, we use the free-particle solution,
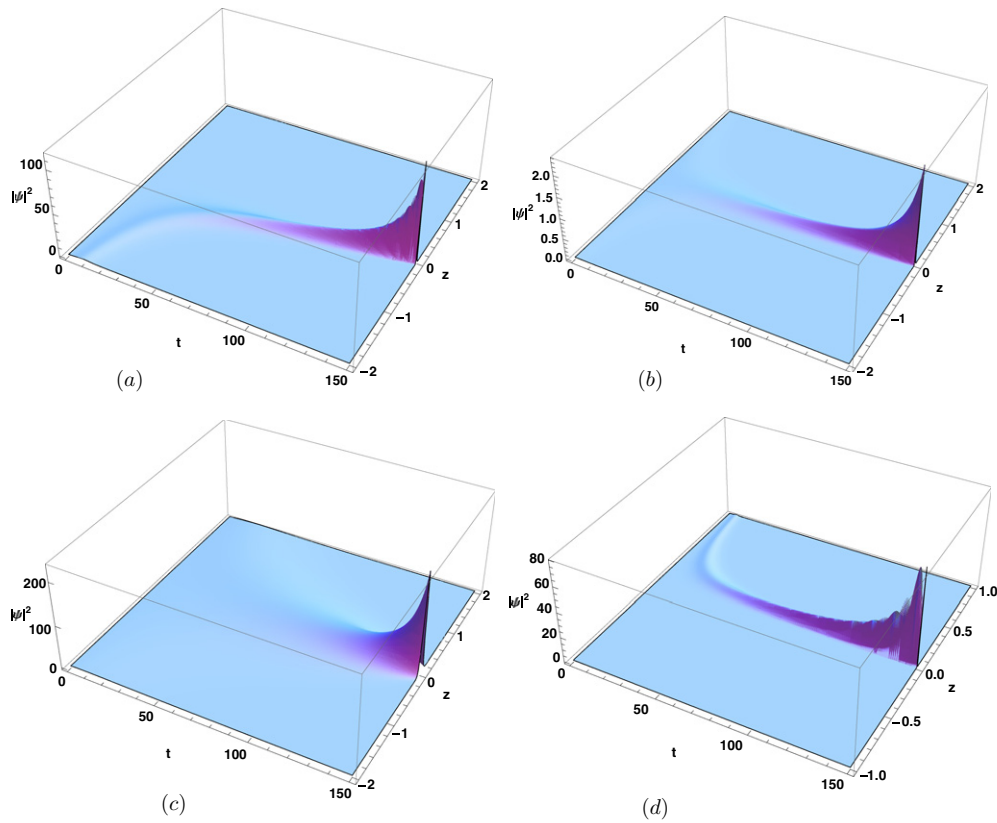
Figure 3. The spatio-temporal dynamics of the bright solitons in the temporally modulated trap with different initial conditions. (*a*) $\gamma_0 = -0.5$, $M_0 = 0.0001$, $A_0 = 0.5$, $l_0 = -5$, $\tau_0 = 1$ and $\lambda' = 5$. (*b*) $\gamma_0 = -0.5$, $M_0 = 0.0001$, $A_0 = 0.01$, $l_0 = 0$, $\tau_0 = 0.02$ and $\lambda' = 5$. (*c*) $\gamma_0 = -0.1$, $M_0 = 0.0001$, $A_0 = 1$, $l_0 = 4$, $\tau_0 = 1$ and $\lambda' = 5$. (d) $\gamma_0 = -1$, $M_0 = 0.0001$, $A_0 = 1$, $l_0 = -3$, $\tau_0 = 1$ and $\lambda' = 1$.

$\phi_1(t) = \cos(\sqrt{M_0}\, t)$, to obtain the bright soliton solution and the expressions for the control parameters. Comparison of the soliton dynamics in these two cases for the same parameter values shows the soliton in the modulated trap to be highly localized and compressed, whereas in the unmodulated trap the soliton moves within the specific interval of time with more or less the same amplitude, undergoing little or no change. We also observe large amplification of the soliton in the former case, when compared to the latter case. In both the cases, we observed that smaller the value of $\sqrt{M_0}$, bigger will be the range of t in which the nonlinearity is continuous, being either attractive or repulsive and hence longer the life time of the soliton. From the above example, it is clear that a temporal modulation of the trap frequency results in solitons which are highly localized and amplified.

In addition, from (17), we can see that though the Khon mode oscillates with the trap frequency as predicted by Kohn's theorem, we find that its amplitude decreases with time. In contrast to this, the Khon mode in an unmodulated trap oscillates with the trap frequency, with constant amplitude.

We would like to point out that the location of the soliton in the temporally modulated traps can be varied by changing l_0 in (17) and the amplitude of the soliton can be controlled
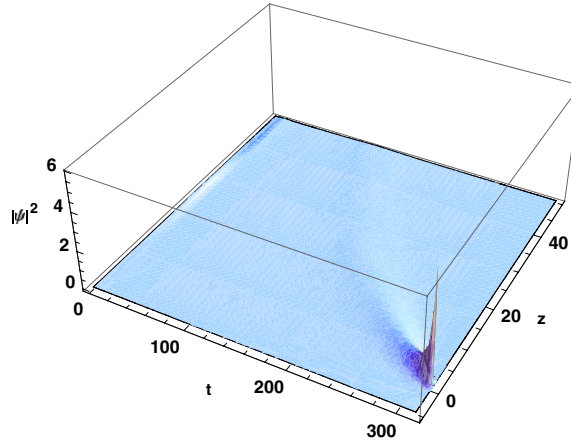
Figure 4. The evolution of the bright solitons when the trap frequency is modulated using (18). The parameter values are $\gamma_0 = -0.5$, $M_0 = 0.001$, $A_0 = 1$, $l_0 = 5$, $\tau_0 = 1$ and $\lambda = 1$.

by varying A_0 in (15) and also the parameters M_0 and λ'. Thus, one can control the dynamics of the soliton completely. Though we have a control over the amplification, we point out here that the amplification of the soliton is bounded by the constraint $\frac{a_s a_\perp N}{a_B}|\tilde{\psi}(z,t)|^2 \ll 1$, where $\tilde{\psi}(z,t) = \sqrt{N}\psi(z,t)$, which is the condition on the solution of the NLSE in the weak interacting limit described above [40]. Hence for the given trap parameters, the solitons cannot be amplified beyond a certain value obtained from the above condition.

In figure 3, we show the evolution of the bright solitons in a modulated trap with different initial conditions. The parameter values chosen are in accordance with the bounds prescribed by the conditions on amplitude and $\gamma(t)$ described above. We see that, for different initial conditions, the solitons build up and show clear localization.

We also point out here that in one dimension the bright solitons themselves are the condensates and in addition the solution (7) is the secant hyperbolic function, which is the well-known fundamental solitonic solution. This solution is self-similar and retains its shape for all values of t in the interval considered. Thus, the solitons in this case are intrinsically stable. Thus, we see that an oscillating temporal modulation, which vanishes asymptotically, leads to localized solitons with huge amplification compared to the solitons created in a trap with constant frequency. Next, we have considered a similar temporal modulation given by

$$V_2(t, \lambda') = \frac{32M_0 \sin^4(\sqrt{M_0}\,t)}{D_1(t)^2} - \frac{8M_0 \sin^2(2\sqrt{M_0}\,t)}{D_1(t)} \tag{18}$$

for $t > 0$. Here, $D_1(t) = 2\sqrt{M_0}\,t - \sin(2\sqrt{M_0}\,t) + 4\sqrt{M_0}\lambda'$, $\sqrt{M_0} > 0$ and $\lambda > 0$. From the above equation, it can be seen that this is also an oscillating potential which vanishes at infinity. The square integrable solution of (11) with the above $V(t)$ is

$$\tilde{\phi}_2(t, \lambda') = \frac{4\sqrt{M_0}\,\sin(\sqrt{M_0}\,t)}{D_1(t)} \tag{19}$$

with eigenvalue $M_0 > 0$. A similar analysis of the soliton dynamics with temporal modulation produced by the above potential was performed. We found that this modulation also leads to the localization of the soliton as shown in figure 4. The Kohn mode behaves in the same way as in the previous case, i.e. it oscillates with the trap frequency with its amplitude decreasing with time.

4. Conclusions

In this paper, we have shown that highly localized and compressed solitons can be created in a BEC confined to a regular harmonic trap, by suitably modulating the trap frequency temporally. We used oscillatory potential functions, which vanish asymptotically to modulate the trap frequencies and have shown that such modulations result in coherent atomic wave packets. We have also discussed how one can control the amplitude, localization and other properties of the solitons by varying the control parameters in detail and have given the soliton profiles for different initial conditions. In addition, we have compared these solitons with those in traps with a constant frequency. We showed that the localization and amplification of the solitons are quite high in the modulated traps showing that such traps are more useful if one wants to generate coherent atomic wave packets, which is the case with atom lasers. We would like to point out here that complex optical potentials with suitable parameters have been used to study BICs in the context of photonic crystals [46, 47] and radiation continuum [48]. In this light, the modulations used in this paper seem more probable physically, especially with the significant development of the techniques which enable us to engineer optical potentials of required parameters in the laboratory. Thus, the use of such temporally modulated traps might be more useful in the construction of atom lasers.

We emphasize here that by effectively combining the solutions of a linear Schrödinger equation with the solutions of the NLSE, we have obtained analytical solutions of the GP equation. This gave us the freedom to introduce temporal modulations in the trap frequency and study the spatio-temporal dynamics of the solitons. In addition, we could study the effect of the modulation on the COM motion using one of the consistency conditions obtained. We have found that due to the modulation, though the COM oscillates with the trap frequency, it loses its amplitude with time.

Acknowledgments

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